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THERMAL STRESSES IN AN ELASTIC SPHERE CONTAINING CONICAL CUTS

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An axisymmetric thermoelastic problem for a sphere with conical cuts in the poles under the action of a quasi-stationary temperature field depending on the meridional angle and on the sphere radius is solved by a method based on the Castigliano variational principle. Orthonormalized systems of polynomials are used as the coordinate functions. Results of numerical calculations of the stress state of a spherical solid are presented.

Key words: *equilibrium equations, boundary conditions, stress functions, potential strain energy.*

The thermoelastic state of a thick-walled sphere was studied in a number of papers (see, e.g., [1–3]). Lur'e [1] derived expressions for stresses and displacements in the case of the action of volume forces whose potential is a harmonic function of the sphere radius. Boley and Weiner [2] considered a similar problem and used the solution obtained for the case of a spherical solid. Kovalenko [3] studied the stress-strain state of a spherical solid under the action of an arbitrary temperature field depending on the radius and meridional angle. The object studied in those papers, however, was a spherical thick-walled solid with no conical cuts.

The present study is a continuation of the paper [4] where the thermoelastic problem was solved for a hollow spherical solid with one conical cut.

Let an axisymmetric temperature field $T = T(r, \theta)$ be applied to a thick-walled sphere with two conical cuts at its poles (Fig. 1). The task is to find the stress state of the solid in the absence of surface and volume loads. The problem includes the equilibrium equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\Big[\frac{\partial \tau}{\partial \theta} + 2\sigma_r - \sigma_\theta - \sigma_\varphi + \tau_{r\theta} \cot\theta\Big] = 0,$$

$$\frac{\partial \tau}{\partial r} + \frac{1}{r}\Big[\frac{\partial \sigma_\theta}{\partial \theta} + (\sigma_\theta - \sigma_\varphi)\cot\theta + 3\tau_{r\theta}\Big] = 0,$$

the equations of continuity of strains

$$r\,\frac{\partial \varepsilon_\varphi}{\partial r} + \cot\theta\Big[\frac{\partial \varepsilon_\varphi}{\partial \theta} - \cot\theta\,(\varepsilon_\theta - \varepsilon_\varphi) - \gamma_{r\theta}\Big] - \varepsilon_r + \varepsilon_\varphi = 0,$$

$$\frac{\partial^2 (r\varepsilon_\varphi)}{\partial r^2} - \frac{\partial \varepsilon_r}{\partial r} + \frac{\cot\theta}{r}\Big[\frac{\partial \varepsilon_r}{\partial \theta} - \frac{\partial (r\gamma_{r\theta})}{\partial r}\Big] = 0,$$

and the boundary conditions

$$\sigma_r\Big|_{r=a} = \sigma_r\Big|_{r=b} = \tau_{r\theta}\Big|_{r=a} = \tau_{r\theta}\Big|_{r=b} = 0, \qquad \sigma_\theta\Big|_{\theta=\theta_1} = \sigma_\theta\Big|_{\theta=\theta_2} = \tau_{r\theta}\Big|_{\theta=\theta_1} = \tau_{r\theta}\Big|_{\theta=\theta_2} = 0.$$

Here, σ_r, σ_φ, σ_θ, $\tau_{r\theta}$, ε_r, ε_φ, ε_θ, and $\gamma_{r\theta}$ are the radial, circumferential, meridional, and shear stresses and strains, which are related, in accordance with Hooke's law, by the formulas

$$\varepsilon_r = [\sigma_r - \nu(\sigma_\varphi + \sigma_\theta)]/E + \alpha T, \qquad \varepsilon_\varphi = [\sigma_\varphi - \nu(\sigma_r + \sigma_\theta)]/E + \alpha T,$$

$$\varepsilon_\theta = [\sigma_\theta - \nu(\sigma_\varphi + \sigma_r)]/E + \alpha T, \qquad \gamma_{r\theta} = 2(1+\nu)\tau_{r\theta}/E,$$

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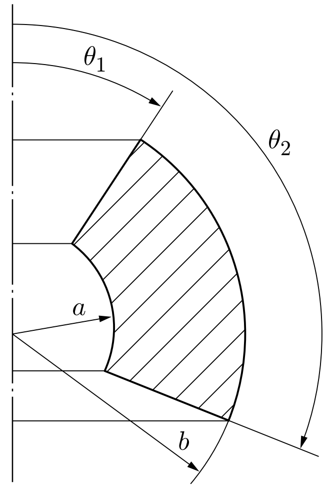

Fig. 1. Geometry and size of the sphere.

where α, ν, and E are the thermal expansion coefficient, Poisson's ratio, and Young's modulus, which are assumed to depend on temperature and, hence, on the coordinates r and θ.

Using the Castigliano variational principle, we find the stress state of the examined solid, such that the potential strain energy acquires the minimum value, i.e., $\delta V = 0$, and the strain continuity equations are valid.

The potential energy is described by the expression

$$V = \int_a^b \int_{\theta_1}^{\theta_2} \Big[\frac{1}{2E}(\sigma_r^2 + \sigma_\varphi^2 + \sigma_\theta^2) - \frac{\nu}{E}(\sigma_r\sigma_\varphi + \sigma_r\sigma_\theta + \sigma_\varphi\sigma_\theta) + \frac{1+\nu}{E}\tau_{r\theta}^2 + \alpha T(\sigma_r + \sigma_\varphi + \sigma_\theta)\Big] r^2 \sin\theta \, d\theta \, dr. \qquad (1)$$

As the sought functions of stresses, we use one of the solutions of the equilibrium equations [5]

$$\sigma_r = \frac{1}{r^3}\Big[2rW_1 + \Big(\frac{\partial W_2}{\partial \theta} + \cot\theta W_2\Big)\Big], \qquad \sigma_\varphi = \frac{1}{r^2}\Big(-\frac{\partial^2 W_1}{\partial \theta^2} + r\frac{\partial W_1}{\partial r} + \frac{\partial^2 W}{\partial r\,\partial \theta}\Big),$$

$$\sigma_\theta = \frac{1}{r^2}\Big(-\cot\theta\frac{\partial W_1}{\partial \theta} + 2r\frac{\partial W_1}{\partial r} + \cot\theta\frac{\partial W_2}{\partial r}\Big), \qquad \tau_{r\theta} = \frac{1}{r^3}\Big(-r\frac{\partial W_1}{\partial \theta} + W_2\Big),$$

where W_1 and W_2 are arbitrary functions of r and θ chosen to satisfy the zero boundary conditions on the surfaces $r = a$ and $r = b$ for the stresses σ_r and $\tau_{r\theta}$ and on the surfaces $\theta = \theta_1$ and $\theta = \theta_2$ for the stresses σ_θ and $\tau_{r\theta}$. The functions W_1 and W_2 are taken in the form

$$W_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^1 F_{1m}(\theta) R_{1n}(r), \qquad W_2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}^2 F_{2m}(\theta) R_{2n}(r),$$

where $F_{1m}(\theta)$, $F_{2m}(\theta)$, $R_{1n}(r)$, and $R_{2n}(r)$ are the coordinate functions satisfying the boundary conditions

$$F_{1m}(\theta_1) = F_{1m}(\theta_2) = F'_{1m}(\theta_1) = F'_{1m}(\theta_2) = 0, \qquad F_{2m}(\theta_1) = F_{2m}(\theta_2) = 0,$$

$$R_{1n}(a) = R_{1n}(b) = 0, \qquad R_{2n}(a) = R_{2n}(b) = 0;$$

A_{mn}^1 and A_{mn}^2 are constants determined below; the prime indicates the derivative. For the coordinate functions $F_{1m}(\theta)$, we use orthonormalized polynomials of the form

$$F_{1m}(\theta) = \sqrt{\frac{2^7 m!(2m+9)}{\pi(m+9)!}}\,\Gamma\Big(\frac{9}{2}\Big)\Big[\Big(\frac{2\theta - \theta_1 - \theta_2}{\theta_2 - \theta_1}\Big)^2 - 1\Big]^2 C_m^{9/2}\Big(\frac{2\theta - \theta_1 - \theta_2}{\theta_2 - \theta_1}\Big);$$

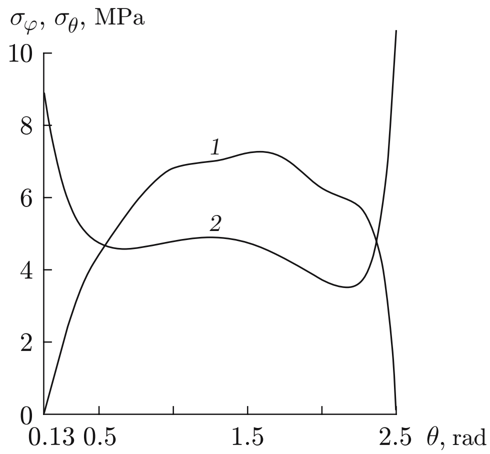

Fig. 2

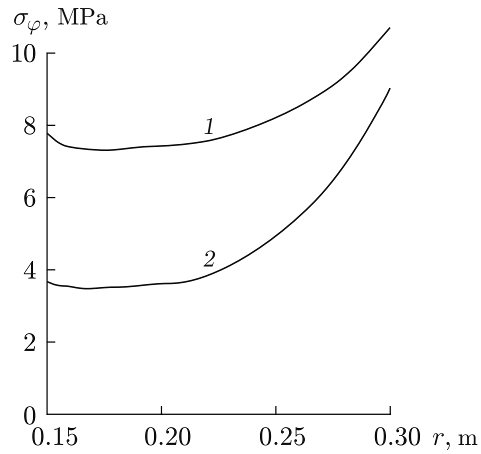

Fig. 3

Fig. 2. Meridionalstress σ_θ (1) and circumferential stress σ_φ (2) versus the angle θ at $r = b$.

Fig. 3. Circumferential stress σ_φ versus the radius r at $\theta = \theta_2$ (1) and $\theta = \theta_1$ (2).

for the coordinate functions $F_{2m}(\theta)$, $R_{1n}(r)$, and $R_{2n}(r)$, we use orthonormalized polynomials of the form

$$F_{2m}(\theta) = \sqrt{\frac{2^7 m!(2m+5)}{\pi(m+4)!}}\,\Gamma\Big(\frac{5}{2}\Big)\Big[\Big(\frac{2\theta-\theta_1-\theta_2}{\theta_2-\theta_1}\Big)^2 - 1\Big] C_m^{5/2}\Big(\frac{2\theta-\theta_1-\theta_2}{\theta_2-\theta_1}\Big),$$

$$R_{1n}(r) = R_{2n}(r) = \sqrt{\frac{2^7 n!(2n+5)}{\pi(n+4)!}}\,\Gamma\Big(\frac{5}{2}\Big)\Big[\Big(\frac{2r-a-b}{b-a}\Big)^2 - 1\Big] C_n^{5/2}\Big(\frac{2r-a-b}{b-a}\Big),$$

where $C_k^{9/2}(x)$ and $C_k^{5/2}(x)$ are the Gegenbauer polynomials [6] of power k and orders 9/2 and 5/2.

By varying the potential energy, i.e., differentiating Eq. (1) with respect to A^1_{mn} and A^2_{mn}, we obtain an infinite system of linear algebraic equations

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a^{11}_{ijmn}A^1_{mn} + a^{12}_{ijmn}A^2_{mn}) = d^1_{ij}, \qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a^{21}_{ijmn}A^1_{mn} + a^{22}_{ijmn}A^2_{mn}) = d^2_{ij}. \tag{2}$$

In these equations,

$$d^k_{ij} = \int_a^b\int_{\theta_1}^{\theta_2} \alpha T(r,\theta)(u^{kr}_{ij} + u^{k\varphi}_{ij} + u^{k\theta}_{ij}) r^2 \sin\theta\, dr\, d\theta,$$

$$a^{ks}_{ijmn} = \int_a^b\int_{\theta_1}^{\theta_2} \Big\{\frac{1}{E}\,(u^{kr}_{ij}u^{sr}_{mn} + u^{k\varphi}_{ij}u^{s\varphi}_{mn} + u^{k\theta}_{ij}u^{s\theta}_{mn})$$

$$-\frac{\nu}{E}\,[u^{kr}_{ij}(u^{s\varphi}_{mn} + u^{s\theta}_{mn}) + u^{k\varphi}_{ij}(u^{s\theta}_{mn} + u^{sr}_{mn}) + u^{k\theta}_{ij}(u^{sr}_{mn} + u^{s\varphi}_{mn})] + \frac{2(1+\nu)}{E}\,u^{k\tau}_{ij}u^{s\tau}_{mn}\Big\} r^2\sin\theta\, dr\, d\theta,$$

$$u^{1r}_{mn} = \frac{2F_{1m}R_{1n}}{r^2}, \quad u^{2r}_{mn} = \frac{F'_{2m}R_{2n} + \cot\,\theta F_{2m}R_{2n}}{r^3}, \quad u^{1\varphi}_{mn} = \frac{rF_{1m}R'_{1n} - F''_{1m}R_{1n}}{r^2},$$

$$u^{2\varphi}_{mn} = \frac{F'_{2m}R'_{2n}}{r^2}, \quad u^{1\theta}_{mn} = \frac{rF_{1m}R'_{1n} - \cot\,\theta F'_{1m}R_{1n}}{r^2}, \quad u^{2\theta}_{mn} = \frac{\cot\,\theta F_{2m}R'_{2n}}{r^2},$$

$$u^{1\tau}_{mn} = -\frac{F'_{1m}R_{1n}}{r^2}, \quad u^{2\tau}_{mn} = \frac{F_{2m}R_{2n}}{r^3}, \qquad k,s = 1,2, \quad i,j,m,n = 0,1,2,\ldots.$$

Equations (2) were solved with the following initial data: $a = 0.15$ m, $b = 0.30$ m, $\theta_1 = 0.13$ rad, $\theta_2 = 2.5$ rad, $\nu = 0.3$, $E = 10^4$ MPa, $\alpha = 7.3 \cdot 10^{-5}$ K^{-1}, and $T(r) = -0.213r^2 + 7.2r - 255$ K. An analysis of the results of numerical experiments shows that it is sufficient to take $m = n = 18$ in series (2) to obtain an approximate solution of the problem considered. In this case, the order of system (2) is $2 \times m \times n = 648$.

Figures 2 and 3 show the circumferential σ_φ and meridional σ_θ stresses as functions of the radius r and the angle θ.

It is seen in Fig. 2 that the circumferential stresses σ_φ at $r = b$ reach the maximum values on the boundaries $\theta = \theta_1$ and $\theta = \theta_2$. This agrees with the results of Litvinov and Mel'nikov [4] and can be attributed to a more significant deformation of the solid owing to sphere bending on the boundaries $\theta = \theta_1$ and $\theta = \theta_2$. The meridional stresses σ_θ are equal to zero on these boundaries.

Figure 3 shows the circumferential stress σ_φ as a function of the radius r at $\theta = \theta_1$ and $\theta = \theta_2$. For all values of r, the stresses at $\theta = \theta_2$ are higher than those at $\theta = \theta_1$. The reason is the greater bending of the sphere at $\theta = \theta_2$ than at $\theta = \theta_1$, owing to the greater freedom of shell motion.

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